

On the deformations and derivations of n -ary multiplicative Hom-Nambu-Lie superalgebras

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Abstract

In this paper, we introduce the relevant concepts of n -ary multiplicative Hom-Nambu-Lie superalgebras and construct three classes of n -ary multiplicative Hom-Nambu-Lie superalgebras. As a generalization of the notion of derivations for n -ary multiplicative Hom-Nambu-Lie algebras, we discuss the derivations of n -ary multiplicative Hom-Nambu-Lie superalgebras. In addition, the theory of one parameter formal deformation of n -ary multiplicative Hom-Nambu-Lie superalgebras is developed by choosing a suitable cohomology.

Key words: n -ary Hom-Nambu-Lie superalgebra, derivation, infinitesimal deformation

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1 Introduction

In 1996, the concept of n -Lie superalgebras was firstly introduced by Y. Daletskii and V. Kushnirenich in [8]. Moreover, N. Cantarini and V. G. Kac gave a more general concept of n -Lie superalgebras again in 2010 in [7]. n -Lie superalgebras are more general structures including n -Lie algebras (n -ary Nambu-Lie algebras), n -ary Nambu-Lie superalgebras and Lie superalgebras.

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The general Hom-algebra structures arose first in connection with quasi-deformation and discretizations of Lie algebras of vector fields. These quasi-deformations lead to quasi-Lie algebras, a generalized Lie algebra structure in which the skewsymmetry and Jacobi conditions are twisted. Hom-Lie algebras, Hom-associative algebras, Hom-Lie superalgebras, Hom-bialgebras, n -ary Hom-Nambu-Lie algebras and quasi-Hom-Lie algebras are discussed in [1–3, 6, 17–20, 24–26]. Generalizations of n -ary algebras of Lie type and associative type by twisting the identities using linear maps have been introduced in [5].

The mathematical theory of deformations has proved to be a powerful tool in modeling physical reality. For example, (algebras associated with) classical quantum mechanics (and field theory) on a Poisson phase space can be deformed to (algebras associated with) quantum mechanics (and quantum field theory). The deformation of algebraic systems has been one of the problems that many mathematical researchers are interested in, Gerstenhaber studied the deformation theory of algebras in a series of papers [10–14]. For example, it has been extended to covariant functors from a small category to algebras. In [15] and [9], it is respectively extended to algebra systems, bialgebras, Hopf algebras, Leibniz pairs and Poisson algebras, etc. In [11], Gerstenhaber developed a theory of deformation of associative and Lie algebras. His theory links cohomologies of these algebras and the Gerstenhaber bracket giving obstructions to deformations. Nijenhuis and Richardson noticed strong similarities between Gerstenhabers theory and the deformations of complex analytic structures on compact manifolds [21]. They axiomatized the theory of deformations via the introduction of graded Lie algebras [22]. One such example was given by the theory of deformations of homomorphisms [23]. Inspired by these works, we study the deformation theory of n -ary multiplicative Hom-Nambu-Lie superalgebras in this paper. In addition, the paper also discusses derivations of n -ary multiplicative Hom-Nambu-Lie superalgebras as a generalization of the notions of derivations for n -ary multiplicative Hom-Nambu-Lie algebras.

This paper is organized as follows. In section 1, we introduce the relevant concepts of n -ary multiplicative Hom-Nambu-Lie superalgebras and construct three classes of n -ary multiplicative Hom-Nambu-Lie superalgebras. In section 2, the notion of derivation introduced for n -ary multiplicative Hom-Nambu-Lie algebras in [2] is extended to n -ary multiplicative Hom-Nambu-Lie superalgebras. In section 3, the theory of deformations of n -ary multiplicative Hom-Nambu-Lie superalgebras is developed by choosing a suitable cohomology.

Definition 1.1. ^[4] An n -ary Nambu-Lie superalgebra is a pair $(\mathfrak{g}, [\cdot, \dots, \cdot])$ consisting of a \mathbb{Z}_2 -graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and a multilinear mapping $[\cdot, \dots, \cdot] : \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_n \rightarrow \mathfrak{g}$, satisfying

$$|[x_1, \dots, x_n]| = |x_1| + \dots + |x_n|,$$

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -(-1)^{|x_i||x_{i+1}|} [x_1, \dots, x_{i+1}, x_i, \dots, x_n],$$

$$[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = \sum_{i=1}^n (-1)^{(|x_1|+\dots+|x_{n-1}|)(|y_1|+\dots+|y_{i-1}|)} \\ \cdot [y_1, \dots, y_{i-1}, [x_1, \dots, x_{n-1}, y_i], y_{i+1}, \dots, y_n],$$

where $|x| \in \mathbb{Z}_2$ denotes the degree of a homogeneous element $x \in \mathfrak{g}$.

Definition 1.2. An n -ary Hom-Nambu-Lie superalgebra is a triple $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ consisting of a \mathbb{Z}_2 -graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, a multilinear mapping $[\cdot, \dots, \cdot] : \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_n \rightarrow \mathfrak{g}$ and a family $\alpha = (\alpha_i)_{1 \leq i \leq n-1}$ of even linear maps $\alpha_i : \mathfrak{g} \rightarrow \mathfrak{g}$, satisfying

$$|[x_1, \dots, x_n]| = |x_1| + \dots + |x_n|, \quad (1.1)$$

$$[x_1, \dots, x_i, x_{i+1}, \dots, x_n] = -(-1)^{|x_i||x_{i+1}|} [x_1, \dots, x_{i+1}, x_i, \dots, x_n], \quad (1.2)$$

$$[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [y_1, \dots, y_n]] = \sum_{i=1}^n (-1)^{(|x_1|+\dots+|x_{n-1}|)(|y_1|+\dots+|y_{i-1}|)} \\ \cdot [\alpha_1(y_1), \dots, \alpha_{i-1}(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \dots, \alpha_{n-1}(y_n)], \quad (1.3)$$

where $|x| \in \mathbb{Z}_2$ denotes the degree of a homogeneous element $x \in \mathfrak{g}$.

An n -ary Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ is multiplicative, if $\alpha = (\alpha_i)_{1 \leq i \leq n-1}$ with $\alpha_1 = \dots = \alpha_{n-1} = \alpha$ and satisfying

$$\alpha[x_1, \dots, x_n] = [\alpha(x_1), \dots, \alpha(x_n)], \forall x_1, x_2, \dots, x_n \in \mathfrak{g}.$$

If the n -ary Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ is multiplicative, then the equation (1.3) can be read:

$$[\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]] = \sum_{i=1}^n (-1)^{(|x_1|+\dots+|x_{n-1}|)(|y_1|+\dots+|y_{i-1}|)} \\ \cdot [\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)]. \quad (1.3')$$

It is clear that n -ary Hom-Nambu-Lie algebras and Hom-Lie superalgebras are particular cases of n -ary Hom-Nambu-Lie superalgebras. In the sequel, when the notation “ $|x|$ ” appears, it means that x is a homogeneous element of degree $|x|$.

Definition 1.3. Let $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ and $(\mathfrak{g}', [\cdot, \dots, \cdot]', \alpha')$ be two n -ary Hom-Nambu-Lie superalgebras, where $\alpha = (\alpha_i)_{1 \leq i \leq n-1}$ and $\alpha' = (\alpha'_i)_{1 \leq i \leq n-1}$. A linear map $f : \mathfrak{g} \rightarrow \mathfrak{g}'$ is an n -ary Hom-Nambu-Lie superalgebra morphism if it satisfies

$$f[x_1, \dots, x_n] = [f(x_1), \dots, f(x_n)]',$$

$$f \circ \alpha_i = \alpha'_i \circ f, \forall i = 1, \dots, n-1.$$

Theorem 1.4. Let $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ be an n -ary multiplicative Hom-Nambu-Lie superalgebra and let $\beta : \mathfrak{g} \rightarrow \mathfrak{g}$ be a morphism of g such that $\beta \circ \alpha = \alpha \circ \beta$. Then $(\mathfrak{g}, \beta \circ [\cdot, \dots, \cdot], \beta \circ \alpha)$ is an n -ary multiplicative Hom-Nambu-Lie superalgebra.

Proof. Put $[\cdot, \dots, \cdot]_\beta := \beta \circ [\cdot, \dots, \cdot]$. Then

$$\begin{aligned}
& (\beta \circ \alpha)[x_1, \dots, x_n]_\beta = (\beta \circ \alpha)(\beta[x_1, \dots, x_n]) \\
& = \beta \circ (\alpha \circ \beta)[x_1, \dots, x_n] \\
& = \beta \circ (\beta \circ \alpha)[x_1, \dots, x_n] \\
& = \beta[\beta \circ \alpha(x_1), \dots, \beta \circ \alpha(x_n)] \\
& = [\beta \circ \alpha(x_1), \dots, \beta \circ \alpha(x_n)]_\beta,
\end{aligned}$$

i.e., $\beta \circ \alpha$ is a morphism of g . Moreover, we have

$$\begin{aligned}
& [\beta \circ \alpha(x_1), \dots, \beta \circ \alpha(x_{n-1}), [y_1, \dots, y_n]_\beta]_\beta \\
& = \beta[\beta \circ \alpha(x_1), \dots, \beta \circ \alpha(x_{n-1}), \beta[y_1, \dots, y_n]] \\
& = \beta^2[\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]] \\
& = \beta^2 \left(\sum_{i=1}^n (-1)^{(|x_1|+\dots+|x_{n-1}|)(|y_1|+\dots+|y_{i-1}|)} [\alpha(y_1), \dots, [x_1, \dots, x_{n-1}, y_i], \dots, \alpha(y_n)] \right) \\
& = \sum_{i=1}^n (-1)^{(|x_1|+\dots+|x_{n-1}|)(|y_1|+\dots+|y_{i-1}|)} \beta[\beta \circ \alpha(y_1), \dots, \beta[x_1, \dots, x_{n-1}, y_i], \dots, \beta \circ \alpha(y_n)] \\
& = \sum_{i=1}^n (-1)^{(|x_1|+\dots+|x_{n-1}|)(|y_1|+\dots+|y_{i-1}|)} [\beta \circ \alpha(y_1), \dots, [x_1, \dots, x_{n-1}, y_i]_\beta, \dots, \beta \circ \alpha(y_n)]_\beta.
\end{aligned}$$

Therefore, $(\mathfrak{g}, \beta \circ [\cdot, \dots, \cdot], \beta \circ \alpha)$ is an n -ary multiplicative Hom-Nambu-Lie superalgebra. \square

In particular, we have the following example.

Example 1.5. Let $(\mathfrak{g}, [\cdot, \dots, \cdot])$ be an n -ary Nambu-Lie superalgebra and let $\rho : \mathfrak{g} \rightarrow \mathfrak{g}$ be an n -ary Nambu-Lie superalgebra endomorphism. Then $(\mathfrak{g}, \rho \circ [\cdot, \dots, \cdot], \rho)$ is an n -ary multiplicative Hom-Nambu-Lie superalgebra.

Definition 1.6. Let $(\mathfrak{g}, [\cdot, \dots, \cdot]_\alpha, \alpha)$ be an n -ary Hom-Nambu-Lie superalgebra. A graded subspace $H \subseteq \mathfrak{g}$ is a Hom-subalgebra of $(\mathfrak{g}, [\cdot, \dots, \cdot]_\alpha, \alpha)$ if $\alpha(H) \subseteq H$ and H is closed under the bracket operation $[\cdot, \dots, \cdot]_\alpha$, i.e., $[u_1, \dots, u_n]_\alpha \in H, \forall u_1, \dots, u_n \in H$.

A graded subspace $H \subseteq \mathfrak{g}$ is a Hom-ideal of $(\mathfrak{g}, [\cdot, \dots, \cdot]_\alpha, \alpha)$ if $\alpha(H) \subseteq H$ and $[u_1, u_2, \dots, u_n]_\alpha \in H, \forall u_1 \in H, u_2, \dots, u_n \in \mathfrak{g}$.

Definition 1.7. Let $(\mathfrak{g}_1, [\cdot, \dots, \cdot]_1, \alpha)$ and $(\mathfrak{g}_2, [\cdot, \dots, \cdot]_2, \beta)$ be two n -ary multiplicative Hom-Nambu-Lie superalgebras. Suppose that $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a linear map. $\mathfrak{G}_\phi = \{(x, \phi(x)) | x \in \mathfrak{g}_1\} \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is called as the graph of a linear map $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$.

Proposition 1.8. Given two n -ary multiplicative Hom-Nambu-Lie superalgebras $(\mathfrak{g}_1, [\cdot, \dots, \cdot]_{\mathfrak{g}_1}, \alpha)$ and $(\mathfrak{g}_2, [\cdot, \dots, \cdot]_{\mathfrak{g}_2}, \beta)$, there is an n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, [\cdot, \dots, \cdot]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}, \alpha + \beta)$, where the bilinear map $[\cdot, \dots, \cdot]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} : \wedge^2(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is given by

$$[u_1 + v_1, \dots, u_n + v_n]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} = [u_1, \dots, u_n]_{\mathfrak{g}_1} + [v_1, \dots, v_n]_{\mathfrak{g}_2}, \forall u_i \in \mathfrak{g}_1, v_i \in \mathfrak{g}_2 (i = 1, 2, \dots, n)$$

and the linear map $(\alpha + \beta) : \mathfrak{g}_1 \oplus \mathfrak{g}_2 \rightarrow \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is given by

$$(\alpha + \beta)(u + v) = \alpha(u) + \beta(v), \forall u \in \mathfrak{g}_1, v \in \mathfrak{g}_2.$$

Proof. For any $u_i \in \mathfrak{g}_1, v_i \in \mathfrak{g}_2$, we have

$$\begin{aligned} & [u_1 + v_1, \dots, u_i + v_i, u_{i+1} + v_{i+1}, \dots, u_n + v_n]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} \\ &= [u_1, \dots, u_i, u_{i+1}, \dots, u_n]_{\mathfrak{g}_1} + [v_1, \dots, v_i, v_{i+1}, \dots, v_n]_{\mathfrak{g}_2} \\ &= -(-1)^{|u_i||u_{i+1}|} [u_1, \dots, u_{i+1}, u_i, \dots, u_n]_{\mathfrak{g}_1} - (-1)^{|v_i||v_{i+1}|} [v_1, \dots, v_{i+1}, v_i, \dots, v_n]_{\mathfrak{g}_2} \\ &= -(-1)^{|u_i||u_{i+1}|} ([u_1, \dots, u_i, u_{i+1}, \dots, u_n]_{\mathfrak{g}_1} + [v_1, \dots, v_i, v_{i+1}, \dots, v_n]_{\mathfrak{g}_2}) \\ &= -(-1)^{|u_i||u_{i+1}|} [u_1 + v_1, \dots, u_{i+1} + v_{i+1}, u_i + v_i, \dots, u_n + v_n]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}. \end{aligned}$$

The bracket is obviously supersymmetric. By a direct computation we have

$$\begin{aligned} & [(\alpha + \beta)(u_1 + v_1), \dots, (\alpha + \beta)(u_{n-1} + v_{n-1}), [x_1 + y_1, \dots, x_n + y_n]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} \\ &= [\alpha(u_1) + \beta(v_1), \dots, \alpha(u_{n-1}) + \beta(v_{n-1}), [x_1 + y_1, \dots, x_n + y_n]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} \\ &= [\alpha(u_1), \dots, \alpha(u_{n-1}), [x_1, \dots, x_n]_{\mathfrak{g}_1}]_{\mathfrak{g}_1} + [\beta(v_1), \dots, \beta(v_{n-1}), [y_1, \dots, y_n]_{\mathfrak{g}_2}]_{\mathfrak{g}_2} \\ &= \sum_{i=1}^n (-1)^{(|u_1| + \dots + |u_{n-1}|)(|x_1| + \dots + |x_{i-1}|)} ([\alpha(x_1), \dots, \alpha(x_{i-1}), [u_1, \dots, u_{n-1}, x_i]_{\mathfrak{g}_1}, \\ & \quad \alpha(x_{i+1}), \dots, \alpha(x_n)]_{\mathfrak{g}_1} + [\beta(y_1), \dots, \beta(y_{i-1}), [v_1, \dots, v_{n-1}, y_i]_{\mathfrak{g}_2}, \beta(y_{i+1}), \dots, \beta(y_n)]_{\mathfrak{g}_2}) \\ &= \sum_{i=1}^n (-1)^{(|u_1| + \dots + |u_{n-1}|)(|x_1| + \dots + |x_{i-1}|)} [\alpha(x_1) + \beta(y_1), \dots, \alpha(x_{i-1}) + \beta(y_{i-1}), \\ & \quad [u_1, \dots, u_{n-1}, x_i]_{\mathfrak{g}_1} + [v_1, \dots, v_{n-1}, y_i]_{\mathfrak{g}_2}, \alpha(x_{i+1}) + \beta(y_{i+1}), \dots, \alpha(x_n) + \beta(y_n)]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} \\ &= \sum_{i=1}^n (-1)^{(|u_1| + \dots + |u_{n-1}|)(|x_1| + \dots + |x_{i-1}|)} [(\alpha + \beta)(x_1 + y_1), \dots, (\alpha + \beta)(x_{i-1} + y_{i-1}), \\ & \quad [u_1 + v_1, \dots, u_{n-1} + v_{n-1}, x_i + y_i]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}, (\alpha + \beta)(x_{i+1} + y_{i+1}), \dots, (\alpha + \beta)(x_n + y_n)]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} \\ &= \sum_{i=1}^n (-1)^{(|u_1| + \dots + |u_{n-1}|)(|x_1| + \dots + |x_{i-1}|)} [(\alpha + \beta)(x_1 + y_1), \dots, (\alpha + \beta)(x_{i-1} + y_{i-1}), \\ & \quad [u_1 + v_1, \dots, u_{n-1} + v_{n-1}, x_i + y_i]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}, (\alpha + \beta)(x_{i+1} + y_{i+1}), \dots, (\alpha + \beta)(x_n + y_n)]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}. \end{aligned}$$

□

Proposition 1.9. A linear map $\phi : (\mathfrak{g}_1, [\cdot, \dots, \cdot]_{\mathfrak{g}_1}, \alpha) \rightarrow (\mathfrak{g}_2, [\cdot, \dots, \cdot]_{\mathfrak{g}_2}, \beta)$ is a morphism of n -ary multiplicative Hom-Nambu-Lie superalgebras if and only if the graph $\mathfrak{G}_\phi \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Hom-subalgebra of $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, [\cdot, \dots, \cdot]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}, \alpha + \beta)$.

Proof. Let $\phi : (\mathfrak{g}_1, [\cdot, \dots, \cdot]_{\mathfrak{g}_1}, \alpha) \rightarrow (\mathfrak{g}_2, [\cdot, \dots, \cdot]_{\mathfrak{g}_2}, \beta)$ be a morphism of n -ary multiplicative Hom-Nambu-Lie superalgebras. Then

$$[u_1 + \phi(u_1), \dots, u_n + \phi(u_n)]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}$$

$$\begin{aligned}
&= [u_1, \dots, u_n]_{\mathfrak{g}_1} + [\phi(u_1), \dots, \phi(u_n)]_{\mathfrak{g}_2} \\
&= [u_1, \dots, u_n]_{\mathfrak{g}_1} + \phi[u_1, \dots, u_n]_{\mathfrak{g}_2}.
\end{aligned}$$

Then the graph \mathfrak{G}_ϕ is closed under the bracket operation $[\cdot, \dots, \cdot]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}$. Furthermore, we obtain

$$(\alpha + \beta)(u + \phi(u)) = \alpha(u) + \beta \circ \phi(u) = \alpha(u) + \phi \circ \alpha(u),$$

which implies that $(\alpha + \beta)(\mathfrak{G}_\phi) \subseteq \mathfrak{G}_\phi$. Thus, \mathfrak{G}_ϕ is a Hom-subalgebra of $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, [\cdot, \dots, \cdot]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}, \alpha + \beta)$.

Conversely, if the graph $\mathfrak{G}_\phi \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a Hom-subalgebra of $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, [\cdot, \dots, \cdot]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2}, \alpha + \beta)$, then we have

$$[u_1 + \phi(u_1), \dots, u_n + \phi(u_n)]_{\mathfrak{g}_1 \oplus \mathfrak{g}_2} = [u_1, \dots, u_n]_{\mathfrak{g}_1} + \phi[u_1, \dots, u_n]_{\mathfrak{g}_2},$$

which implies that

$$\phi[u_1, \dots, u_n]_{\mathfrak{g}_2} = \phi[u_1, \dots, u_n]_{\mathfrak{g}_1}.$$

Furthermore, $(\alpha + \beta)(\mathfrak{G}_\phi) \subset \mathfrak{G}_\phi$ yields that

$$(\alpha + \beta)(u + \phi(u)) = \alpha(u) + \beta \circ \phi(u) \in \mathfrak{G}_\phi,$$

which is equivalent to the condition $\beta \circ \phi(u) = \phi \circ \alpha(u)$, i.e. $\beta \circ \phi = \phi \circ \alpha$. Therefore, ϕ is a morphism of n -ary multiplicative Hom-Nambu-Lie superalgebras. \square

2 Derivations of n -ary multiplicative Hom-Nambu-Lie superalgebras

Let $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ be an n -ary multiplicative Hom-Nambu-Lie superalgebra. We denote by α^k the k -times compositions of α . In particular, we set $\alpha^0 = \text{id}$.

Definition 2.1. For $k \geq 0$, we call $D \in \text{End}(\mathfrak{g})$ an α^k -derivation of the n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ if

$$D \circ \alpha = \alpha \circ D$$

and for $x_i \in \mathfrak{g}$ ($i = 1, \dots, n$),

$$D[x_1, \dots, x_n] = \sum_{i=1}^n (-1)^{|D|(|x_1| + \dots + |x_{i-1}|)} [\alpha^k(x_1), \dots, \alpha^k(x_{i-1}), D(x_i), \alpha^k(x_{i+1}), \dots, \alpha^k(x_n)].$$

We denote by $\text{Der}_{\alpha^k}(\mathfrak{g})$ the set of α^k -derivations of the n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$. Notice that we obtain classical derivations for $k = 0$.

For $\mathcal{X} \in \mathfrak{g}^{\wedge^{n-1}}$ satisfying $\alpha(\mathcal{X}) = \mathcal{X}$ and $k \geq 0$, we define the map $\text{ad}_k(\mathcal{X}) \in \text{End}(\mathfrak{g})$ by

$$\text{ad}_k(\mathcal{X})(y) = [x_1, \dots, x_{n-1}, \alpha^k(y)], \forall y \in \mathfrak{g}.$$

Then

Lemma 2.2. *The map $\text{ad}_k(\mathcal{X})$ is an α^{k+1} -derivation and is called an inner α^{k+1} -derivation.*

We denote by $\text{Inn}_{\alpha^k}(\mathfrak{g})$ the \mathbb{K} -vector space generated by all inner α^{k+1} -derivations. For any $D \in \text{Der}_{\alpha^k}(\mathfrak{g})$ and $D' \in \text{Der}_{\alpha^{k'}}(\mathfrak{g})$, we define their commutator $[D, D'] = D \circ D' - (-1)^{|D||D'|} D' \circ D$. Set $\text{Der}(\mathfrak{g}) = \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(\mathfrak{g})$ and $\text{Inn}(\mathfrak{g}) = \bigoplus_{k \geq 0} \text{Inn}_{\alpha^k}(\mathfrak{g})$.

Lemma 2.3. *For any $D \in \text{Der}_{\alpha^k}(\mathfrak{g})$ and $D' \in \text{Der}_{\alpha^{k'}}(\mathfrak{g})$, we have $[D, D'] \in \text{Der}_{\alpha^{k+k'}}(\mathfrak{g})$.*

Proof. Let $x_i \in \mathfrak{g}, 1 \leq i \leq n$. $D \in \text{Der}_{\alpha^k}(\mathfrak{g})$ and $D' \in \text{Der}_{\alpha^{k'}}(\mathfrak{g})$, then

$$\begin{aligned}
& D \circ D'([x_1, \dots, x_n]) \\
&= D\left(\sum_{i=1}^n (-1)^{|D'|(|x_1|+\dots+|x_{i-1}|)} [\alpha^{k'}(x_1), \dots, \alpha^{k'}(x_{i-1}), D'(x_i), \alpha^{k'}(x_{i+1}), \dots, \alpha^{k'}(x_n)]\right) \\
&= \sum_{i=1}^n (-1)^{|D'|(|x_1|+\dots+|x_{i-1}|)} D[\alpha^{k'}(x_1), \dots, \alpha^{k'}(x_{i-1}), D'(x_i), \alpha^{k'}(x_{i+1}), \dots, \alpha^{k'}(x_n)] \\
&= \sum_{i=1}^n (-1)^{|D'|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} \\
&\quad \cdot [\alpha^{k+k'}(x_1), \dots, \alpha^{k+k'}(x_{i-1}), D \circ D'(x_i), \alpha^{k+k'}(x_{i+1}), \dots, \alpha^{k+k'}(x_n)] \\
&+ \sum_{i < j} (-1)^{|D'|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D|(|x_1|+\dots+|x_{j-1}|+|D'|)} \\
&\quad \cdot [\alpha^{k+k'}(x_1), \dots, \alpha^k(D'(x_i)), \dots, \alpha^{k'}(D(x_j)), \dots, \alpha^{k+k'}(x_n)] \\
&+ \sum_{i > j} (-1)^{|D'|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D|(|x_1|+\dots+|x_{j-1}|)} \\
&\quad \cdot [\alpha^{k+k'}(x_1), \dots, \alpha^{k'}(D(x_j)), \dots, \alpha^k(D'(x_i)), \dots, \alpha^{k+k'}(x_n)].
\end{aligned}$$

and

$$\begin{aligned}
& -(-1)^{|D||D'|} D' \circ D([x_1, \dots, x_n]) \\
&= -(-1)^{|D||D'|} D'\left(\sum_{i=1}^n (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} [\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)]\right) \\
&= -(-1)^{|D||D'|} \sum_{i=1}^n (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} D'[\alpha^k(x_1), \dots, D(x_i), \dots, \alpha^k(x_n)] \\
&= -(-1)^{|D||D'|} \sum_{i=1}^n (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D'|(|x_1|+\dots+|x_{i-1}|)} \\
&\quad \cdot [\alpha^{k'+k}(x_1), \dots, \alpha^{k'+k}(x_{i-1}), D' \circ D(x_i), \alpha^{k'+k}(x_{i+1}), \dots, \alpha^{k'+k}(x_n)] \\
&- (-1)^{|D||D'|} \sum_{i < j} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D'|(|x_1|+\dots+|x_{j-1}|+|D|)} \\
&\quad \cdot [\alpha^{k'+k}(x_1), \dots, \alpha^{k'+k}(x_{i-1}), D' \circ D(x_i), \alpha^{k'+k}(x_{i+1}), \dots, \alpha^{k'+k}(x_n)] \\
&- (-1)^{|D||D'|} \sum_{i > j} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D'|(|x_1|+\dots+|x_{j-1}|+|D|)} \\
&\quad \cdot [\alpha^{k'+k}(x_1), \dots, \alpha^{k'+k}(x_{i-1}), D' \circ D(x_i), \alpha^{k'+k}(x_{i+1}), \dots, \alpha^{k'+k}(x_n)]
\end{aligned}$$

$$\begin{aligned}
& \cdot [\alpha^{k'+k}(x_1), \dots, \alpha^{k'}(D(x_i)), \dots, \alpha^k(D'(x_j)), \dots, \alpha^{k'+k}(x_n)] \\
& - (-1)^{|D||D'|} \sum_{i>j} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D'|(|x_1|+\dots+|x_{j-1}|)} \\
& \cdot [\alpha^{k'+k}(x_1), \dots, \alpha^k(D'(x_j)), \dots, \alpha^{k'}(D(x_i)), \dots, \alpha^{k'+k}(x_n)].
\end{aligned}$$

Then we obtain

$$\begin{aligned}
& [D, D']([x_1, \dots, x_n]) = (D \circ D' - (-1)^{|D||D'|} D' \circ D)([x_1, \dots, x_n]) \\
& = \sum_{i=1}^n (-1)^{|D'|(|x_1|+\dots+|x_{i-1}|)} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} [\alpha^{k+k'}(x_1), \dots, \alpha^{k+k'}(x_{i-1}), \\
& \quad (D \circ D' - (-1)^{|D||D'|} D' \circ D)(x_i), \alpha^{k+k'}(x_{i+1}), \dots, \alpha^{k+k'}(x_n)] \\
& = \sum_{i=1}^n (-1)^{(|D'|+|D|)(|x_1|+\dots+|x_{i-1}|)} \\
& \quad \cdot [\alpha^{k+k'}(x_1), \dots, \alpha^{k+k'}(x_{i-1}), [D, D'](x_i), \alpha^{k+k'}(x_{i+1}), \dots, \alpha^{k+k'}(x_n)],
\end{aligned}$$

which yields that $[D, D'] \in \text{Der}_{\alpha^{k+k'}}(\mathfrak{g})$. \square

Proposition 2.4. *The pair $(\text{Der}(\mathfrak{g}), [\cdot, \cdot])$, where the bracket is the usual commutator, defines a Lie superalgebra and $\text{Inn}(\mathfrak{g})$ constitutes an ideal of it.*

Proof. $(\text{Der}(\mathfrak{g}), [\cdot, \cdot])$ is a Lie superalgebra by using Lemma 2.3. We show that $\text{Inn}(\mathfrak{g})$ is an ideal. Let $\text{ad}_{k-1}(\mathcal{X})(y) = [x_1, \dots, x_{n-1}, \alpha^{k-1}(y)]$ be an inner α^k -derivation on \mathfrak{g} and $D \in \text{Der}_{\alpha^{k'}}(\mathfrak{g})$ for $k \geq 1$ and $k' \geq 0$. Then $[D, \text{ad}_{k-1}(\mathcal{X})] \in \text{Der}_{\alpha^{k+k'}}(\mathfrak{g})$ and for any $y \in \mathfrak{g}$

$$\begin{aligned}
& [D, \text{ad}_{k-1}(\mathcal{X})](y) \\
& = D[x_1, \dots, x_{n-1}, \alpha^{k-1}(y)] - (-1)^{|D|(|x_1|+\dots+|x_{n-1}|)} [x_1, \dots, x_{n-1}, \alpha^{k-1}(D(y))] \\
& = D[\alpha^k(x_1), \dots, \alpha^k(x_{n-1}), \alpha^{k-1}(y)] - (-1)^{|D|(|x_1|+\dots+|x_{n-1}|)} \\
& \quad \cdot [\alpha^{k+k'}(x_1), \dots, \alpha^{k+k'}(x_{n-1}), \alpha^{k-1}(D(y))] \\
& = \sum_{i \leq n-1} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} [\alpha^{k+k'}(x_1), \dots, D(\alpha^k(x_i)), \dots, \alpha^{k+k'}(x_{n-1}), \alpha^{k+k'-1}(y)] \\
& = \sum_{i \leq n-1} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} [x_1, \dots, D(x_i), \dots, x_{n-1}, \alpha^{k+k'-1}(y)] \\
& = \sum_{i \leq n-1} (-1)^{|D|(|x_1|+\dots+|x_{i-1}|)} \text{ad}_{k+k'-1}(x_1 \wedge \dots \wedge D(x_i) \wedge \dots \wedge x_{n-1})(y).
\end{aligned}$$

Therefore, $[D, \text{ad}_{k-1}(\mathcal{X})] \in \text{Inn}_{\alpha^{k+k'-1}}(\mathfrak{g})$. \square

3 Deformations of n -ary multiplicative Hom-Nambu-Lie superalgebras

Definition 3.1. ^[16] For $m \geq 1$, we call m -coboundary operator of the n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ the even linear map $\delta^m : C^m(\mathfrak{g}, V) \rightarrow C^{m+1}(\mathfrak{g}, V)$ by

$$\begin{aligned} & (\delta^m f)(\mathcal{X}_1, \dots, \mathcal{X}_m, \mathcal{X}_{m+1}, z) \\ &= \sum_{i < j} (-1)^i (-1)^{|\mathcal{X}_i|(|\mathcal{X}_{i+1}| + \dots + |\mathcal{X}_{j-1}|)} f(\alpha(\mathcal{X}_1), \dots, \widehat{\alpha(\mathcal{X}_i)}, \dots, [\mathcal{X}_i, \mathcal{X}_j]_\alpha, \dots, \alpha(\mathcal{X}_{m+1}), \alpha(z)) \\ &+ \sum_{i=1}^{m+1} (-1)^i (-1)^{|\mathcal{X}_i|(|\mathcal{X}_{i+1}| + \dots + |\mathcal{X}_{m+1}|)} f(\alpha(\mathcal{X}_1), \dots, \widehat{\alpha(\mathcal{X}_i)}, \dots, \alpha(\mathcal{X}_{m+1}), \mathcal{X}_i \cdot z) \\ &+ \sum_{i=1}^{m+1} (-1)^{i+1} (-1)^{|\mathcal{X}_i|(|f| + |\mathcal{X}_1| + \dots + |\mathcal{X}_{i-1}|)} \alpha^m(\mathcal{X}_i) \cdot f(\mathcal{X}_1, \dots, \widehat{\mathcal{X}_i}, \dots, \mathcal{X}_{m+1}, z) \\ &+ (-1)^m (f(\mathcal{X}_1, \dots, \mathcal{X}_m, \quad) \cdot \mathcal{X}_{m+1}) \bullet_\alpha \alpha^m(z), \end{aligned}$$

where $\mathcal{X}_i = \mathcal{X}_i^1 \wedge \dots \wedge \mathcal{X}_i^{n-1} \in \mathfrak{g}^{\wedge^{n-1}}$, $i = 1, \dots, m+1$, $z \in \mathfrak{g}$ and the last term is defined by

$$\begin{aligned} (f(\mathcal{X}_1, \dots, \mathcal{X}_m, \quad) \cdot \mathcal{X}_{m+1}) \bullet_\alpha \alpha^m(z) &= \sum_{i=1}^{n-1} (-1)^{(|f| + |\mathcal{X}_1| + \dots + |\mathcal{X}_m|)(|\mathcal{X}_{m+1}^1| + \dots + |\mathcal{X}_{m+1}^{i-1}|)} \\ &\cdot [\alpha^m(\mathcal{X}_{m+1}^1), \dots, f(\mathcal{X}_1, \dots, \mathcal{X}_m, \mathcal{X}_{m+1}^i), \dots, \alpha^m(\mathcal{X}_{m+1}^{n-1}), \alpha^m(z)]. \end{aligned}$$

Theorem 3.2. ^[16] Let $f \in C^m(\mathfrak{g}, V)$ be an m -cochain. Then $\delta^{m+1} \circ \delta^m(f) = 0$.

In [16], it also points out that, the map $f \in C^m(\mathfrak{g}, V)$ is called an m -supercocycle if $\delta^m f = 0$. We denote by $Z^m(\mathfrak{g}, V)$ the graded subspace spanned by m -supercocycles. Since $\delta^{m+1} \circ \delta^m(f) = 0$ for all $f \in C^m(\mathfrak{g}, V)$, $\delta^{m-1} C^{m-1}(\mathfrak{g}, V)$ is a graded subspace of $Z^m(\mathfrak{g}, V)$. Therefore we can define a graded cohomology space $H^m(\mathfrak{g}, V)$ of \mathfrak{g} as the graded factor space $Z^m(\mathfrak{g}, V) / \delta^{m-1} C^{m-1}(\mathfrak{g}, V)$.

We next will discuss the deformation of n -ary multiplicative Hom-Nambu-Lie superalgebras. Let $\mathbb{K}[[t]]$ denote the power series ring in one variable t with coefficients in \mathbb{K} and $\mathfrak{g}[[t]]$ be the set of formal series whose coefficients are elements of the vector space \mathfrak{g} .

Definition 3.3. Let $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ be an n -ary multiplicative Hom-Nambu-Lie superalgebra over \mathbb{K} . A deformation of $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ is given by $\mathbb{K}[[t]]$ - n -linear map

$$f_t = \sum_{p \geq 0} f_p t^p : \mathfrak{g}[[t]] \times \dots \times \mathfrak{g}[[t]] \rightarrow \mathfrak{g}[[t]]$$

such that $(\mathfrak{g}[[t]], f_t, \alpha)$ is also an n -ary multiplicative Hom-Nambu-Lie superalgebra. We call f_1 the infinitesimal deformation of $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$.

Since $(\mathfrak{g}[[t]], f_t, \alpha)$ is an n -ary multiplicative Hom-Nambu-Lie superalgebra, f_t satisfies

$$\alpha \circ f_t(x_1, \dots, x_n) = f_t(\alpha(x_1), \dots, \alpha(x_n)), \quad (a)$$

$$|f_t(x_1, \dots, x_n)| = |x_1| + \dots + |x_n|, \quad (b)$$

$$\begin{aligned} f_t(\alpha(x_1), \dots, \alpha(x_{n-1}), f_t(y_1, \dots, y_n)) &= \sum_{i=1}^n (-1)^{(|x_1| + \dots + |x_{n-1}|)(|y_1| + \dots + |y_{i-1}|)} \\ &\cdot f_t(\alpha(y_1), \dots, \alpha(y_{i-1}), f_t(x_1, \dots, x_{n-1}, y_i), \alpha(y_{i+1}), \dots, \alpha(y_n)). \end{aligned} \quad (c)$$

(a) – (c) is respectively equivalent to

$$\alpha \circ f_p(x_1, \dots, x_n) = f_p(\alpha(x_1), \dots, \alpha(x_n)), \quad (a')$$

$$|f_p(x_1, \dots, x_n)| = |x_1| + \dots + |x_n|, \quad (b')$$

$$\begin{aligned} \sum_{p+q=l} f_p(\alpha(x_1), \dots, \alpha(x_{n-1}), f_q(y_1, \dots, y_n)) &= \sum_{i=1}^n (-1)^{(|x_1| + \dots + |x_{n-1}|)(|y_1| + \dots + |y_{i-1}|)} \\ &\cdot \left(\sum_{p+q=l} f_p(\alpha(y_1), \dots, \alpha(y_{i-1}), f_q(x_1, \dots, x_{n-1}, y_i), \alpha(y_{i+1}), \dots, \alpha(y_n)) \right). \end{aligned} \quad (c')$$

We call these the deformation equations for an n -ary multiplicative Hom-Nambu-Lie superalgebra.

(a') and (b') shows that $f_p \in C^1(\mathfrak{g}, \mathfrak{g})_{\bar{0}}$. In (c'), set $l = 1$, then

$$\begin{aligned} &[\alpha(x_1), \dots, \alpha(x_{n-1}), f_1(y_1, \dots, y_n)] + f_1(\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]) \\ &- \sum_{i=1}^n (-1)^{(|x_1| + \dots + |x_{n-1}|)(|y_1| + \dots + |y_{i-1}|)} \left([\alpha(y_1), \dots, \alpha(y_{i-1}), f_1(x_1, \dots, x_{n-1}, y_i), \right. \\ &\quad \left. \alpha(y_{i+1}), \dots, \alpha(y_n)] + f_1(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)) \right) \\ &= 0, \end{aligned}$$

i.e., $\delta^1 f_1(x_1, \dots, x_{n-1}, [y_1, \dots, y_{n-1}, y_n]) = 0$. Hence the infinitesimal deformation $f_1 \in Z^1(\mathfrak{g}, \mathfrak{g})_{\bar{0}}$.

Definition 3.4. Two deformations f_t and $f_{t'}$ of the n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ are said to be equivalent, if there exists an isomorphism of n -ary multiplicative Hom-Nambu-Lie superalgebras $\Phi_t : (\mathfrak{g}, f_t, \alpha) \rightarrow (\mathfrak{g}, f_{t'}, \alpha)$, where $\Phi_t = \sum_{i \geq 0} \varphi_i t^i$, $\varphi_i : \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear map such that

$$\varphi_0 = \text{id}_{\mathfrak{g}}; \quad \varphi_i \circ \alpha = \alpha \circ \varphi_i;$$

$$\Phi_t \circ f_t(x_1, \dots, x_n) = f_{t'}(\Phi_t(x_1), \dots, \Phi_t(x_n)),$$

and is denoted by $f_t \sim f_{t'}$. When $f_1 = f_2 = \dots = 0$, $f_1 = f_0$ is called the null deformation; if $f_t \sim f_0$, then f_t is called the trivial deformation.

Theorem 3.5. *Let f_t and f'_t be two equivalent deformations of the n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$. Then the infinitesimal deformations f_1 and f'_1 belong to the same cohomology class in the cohomology group $H^2(\mathfrak{g}, \mathfrak{g})$.*

Proof. Put $B^2(\mathfrak{g}, \mathfrak{g}) := \delta^1 C^1(\mathfrak{g}, \mathfrak{g})$. It is enough to prove that $f_1 - f'_1 \in B^2(\mathfrak{g}, \mathfrak{g})$. Let $\Phi_t : (\mathfrak{g}, f_t, \alpha) \rightarrow (\mathfrak{g}, f'_t, \alpha)$ be an isomorphism of n -ary multiplicative Hom-Nambu-Lie superalgebras. Then $\varphi_1 \in C^1(\mathfrak{g}, \mathfrak{g})_{\bar{0}}$ and

$$\sum_{i \geq 0} \varphi_i \left(\sum_{j \geq 0} f_j(x_1, \dots, x_n) \right) t^{i+j} = \sum_{i \geq 0} f'_i \left(\sum_{j_1 \geq 0} \varphi_{j_1}(x_1), \dots, \sum_{j_n \geq 0} \varphi_{j_n}(x_n) \right) t^{i+j_1+\dots+j_n},$$

comparing with the coefficients of t^1 for two sides of the above equation, we obtain

$$\begin{aligned} f_1(x_1, \dots, x_n) + \varphi_1[x_1, x_2, \dots, x_n] &= [\varphi_1(x_1), x_2, \dots, x_n] + [x_1, \varphi_1(x_2), x_3, \dots, x_n] \\ &\quad + \dots + [x_1, \dots, x_{n-1}, \varphi_1(x_n)] + f'_1(x_1, \dots, x_n). \end{aligned}$$

Furthermore, one gets

$$\begin{aligned} f_1(x_1, \dots, x_n) - f'_1(x_1, \dots, x_n) &= -\varphi_1[x_1, x_2, \dots, x_n] + [\varphi_1(x_1), x_2, \dots, x_n] \\ &\quad + [x_1, \varphi_1(x_2), x_3, \dots, x_n] + \dots + [x_1, \dots, x_{n-1}, \varphi_1(x_n)] \\ &= -\varphi_1[x_1, \dots, x_n] + \sum_{i=1}^n (-1)^{n-i} (-1)^{|x_i|(|x_{i+1}|+\dots+|x_n|)} [x_1, \dots, \widehat{x_i}, \dots, x_n, \varphi_1(x_i)] \\ &= \delta^1 \varphi_1(x_1, \dots, x_n). \end{aligned}$$

Therefore, $f_1 - f'_1 = \delta^1 \varphi_1 \in \delta^1 C^1(\mathfrak{g}, \mathfrak{g})_{\bar{0}}$, i.e., $f_1 - f'_1 \in B^2(\mathfrak{g}, \mathfrak{g})$. \square

An n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ is *analytically rigid* if every deformation f_t is equivalent to the null deformation f_0 . We have a fundamental theorem.

Theorem 3.6. *If $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ is an n -ary multiplicative Hom-Nambu-Lie superalgebra with $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, then $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ is analytically rigid.*

Proof. Let f_t be a deformation of the n -ary multiplicative Hom-Nambu-Lie superalgebra $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ with $f_t = f_0 + f_r t^r + f_{r+1} t^{r+1} + \dots$, i.e., $f_1 = f_2 = \dots = f_{r-1} = 0$. Then set $l = r$ in (c'), we have

$$\begin{aligned} &f_r(\alpha(x_1), \dots, \alpha(x_{n-1}), [y_1, \dots, y_n]) + [\alpha(x_1), \dots, \alpha(x_{n-1}), f_r(y_1, \dots, y_n)] \\ &- \sum_{i=1}^n (-1)^{(|x_1|+\dots+|x_{n-1}|)(|y_1|+\dots+|y_{i-1}|)} \\ &\cdot \left([\alpha(y_1), \dots, \alpha(y_{i-1}), f_r(x_1, \dots, x_{n-1}, y_i), \alpha(y_{i+1}), \dots, \alpha(y_n)] \right. \\ &\left. + f_r(\alpha(y_1), \dots, \alpha(y_{i-1}), [x_1, \dots, x_{n-1}, y_i], \alpha(y_{i+1}), \dots, \alpha(y_n)) \right) \end{aligned}$$

$$=0,$$

i.e, $\delta^2 f_r(x_1, \dots, x_{n-1}, [y_1, \dots, y_n]) = 0$, $\delta^2(f_r) = 0$, that is, $f_r \in Z^2(\mathfrak{g}, \mathfrak{g})_{\bar{0}}$. By our assumption $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, one gets $f_r \in B^2(\mathfrak{g}, \mathfrak{g})_{\bar{0}}$, thus we can find $h_r \in C^1(\mathfrak{g}, \mathfrak{g})_{\bar{0}}$ such that $f_r = \delta^1 h_r$. Put $\Phi_t = \text{id}_{\mathfrak{g}} - h_r t^r$, then $\Phi_t \circ (\text{id}_{\mathfrak{g}} + h_r t^r + h_r^2 t^{2r} + h_r^3 t^{3r} + \dots) = (\text{id}_{\mathfrak{g}} - h_r t^r) \circ (\text{id}_{\mathfrak{g}} + h_r t^r + h_r^2 t^{2r} + h_r^3 t^{3r} + \dots) = (\text{id}_{\mathfrak{g}} + h_r t^r + h_r^2 t^{2r} + h_r^3 t^{3r} + \dots) - (h_r t^r + h_r^2 t^{2r} + h_r^3 t^{3r} + \dots) = \text{id}_{\mathfrak{g}}$, moreover, $(\text{id}_{\mathfrak{g}} + h_r t^r + h_r^2 t^{2r} + h_r^3 t^{3r} + \dots) \circ \Phi_t = \text{id}_{\mathfrak{g}}$. Hence $\Phi_t : \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear isomorphism and $\Phi_t \circ \alpha = \alpha \circ \Phi_t$. Set $f'_t(x_1, \dots, x_n) = \Phi_t^{-1} f_t(\Phi_t(x_1), \dots, \Phi_t(x_n))$, then f'_t is also a deformation of $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ and $f_t \sim f'_t$. Note that $\Phi_t f'_t(x_1, \dots, x_n) = f_t(\Phi_t(x_1), \dots, \Phi_t(x_n))$. Let $f'_t = \sum_{i \geq 0} f'_i t^i$. Then

$$(\text{id}_{\mathfrak{g}} - h_r t^r) \sum_{i \geq 0} f'_i(x_1, \dots, x_n) t^i = (f_0 + \sum_{i \geq r} f_i t^i)(x_1 - h_r(x_1) t^r, \dots, x_n - h_r(x_n) t^r).$$

So

$$\begin{aligned} & \sum_{i \geq 0} f'_i(x_1, \dots, x_n) t^i - \sum_{i \geq 0} h_r \circ f'_i(x_1, \dots, x_n) t^{i+r} \\ &= f_0(x_1, \dots, x_n) - \sum_{i=1}^n f_0(x_1, \dots, h_r(x_i), \dots, x_n) t^r \\ &+ \sum_{1 \leq i < j \leq n} f_0(x_1, \dots, h_r(x_i), \dots, h_r(x_j), \dots, x_n) t^{2r} \\ &- \sum_{1 \leq i < j < k \leq n} f_0(x_1, \dots, h_r(x_i), \dots, h_r(x_j), \dots, h_r(x_k), \dots, x_n) t^{3r} + \dots \\ &+ (-1)^n f_0(h_r(x_1), h_r(x_2), \dots, h_r(x_n)) t^{nr} + \sum_{i \geq r} f_i(x_1, \dots, x_n) t^i \\ &- \sum_{i \geq r} \sum_{j=1}^n f_i(x_1, \dots, h_r(x_j), \dots, x_n) t^{i+r} \\ &+ \sum_{i \geq r} \sum_{1 \leq j < k \leq n} f_i(x_1, \dots, h_r(x_j), \dots, h_r(x_k), \dots, x_n) t^{i+2r} + \dots \end{aligned}$$

By the above equation, one gets

$$\begin{aligned} f'_0(x_1, \dots, x_n) &= f_0(x_1, \dots, x_n) = [x_1, \dots, x_n]; \\ f'_1(x_1, \dots, x_n) &= \dots = f'_{r-1}(x_1, \dots, x_n) = 0; \\ f'_r(x_1, \dots, x_n) - h_r[x_1, \dots, x_n] &= - \sum_{i=1}^n [x_1, \dots, h_r(x_i), \dots, x_n] + f_r(x_1, \dots, x_n). \end{aligned}$$

Furthermore, we have

$$f'_r(x_1, \dots, x_n) = -\delta^1 h_r(x_1, \dots, x_n) + f_r(x_1, \dots, x_n) = 0,$$

hence, $f'_t = f_0 + \sum_{i \geq r+1} f'_i t^i$. By induction, one can prove $f_t \sim f_0$, that is, $(\mathfrak{g}, [\cdot, \dots, \cdot], \alpha)$ is analytically rigid. \square

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